# SOME EXACT SOLUTIONS OF THE EQUATIONS OF A STATIONARY MONOENERGETIC BEAM OF CHARGED PARTICLES

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We consider the class of invariant solutions which can describe only vortex flows (curl P ≠ 0, P is the generalized momentum) and show that they contain solutions corresponding to flows from a plane or cylindrical emitter with a linear voltage drop across it (direct heating) in the temperature-limited regime*. The solution is obtained in analytic form for emission from a plane in a uniform magnetic field perpendicular to the flow plane. It also (for $\beta = 0$) defines a plane magnetron in the T-regime. The solution of the problem for a cylindrical emitter reduces to considering equations describing a cylindrical diode or magnetron in the T-regime, where the shape of the collector is given by the potential distribution curve for these cases. We can extend the results to a relativistic beam if restrictions are imposed on its relative dimensions which permit us to ignore the magnetic self-field. Brillouin type flows (including irrotational ones) are studied in which particles move without intersecting the equipotential surfaces along three-dimensional spirals on the surface of cones. An analytic solution is given for relativistic Brillouin flow in a conical diode when strict allowance is made for the magnetic self-field.

§1. Below we shall study solutions of the equations of a stationary monoenergetic beam of charged particles with the same magnitude and sign of the charge-mass ratio $\eta$ in the nonrelativistic case [1, 2]; their form is given in Table 1.

Table 1

| | $\xi$ | $\mathbf{V}$ | $\varphi$ | $\rho$ | $\mathbf{H}$ |
|---|---|---|---|---|---|
| 1° | $x$ | $\mathbf{J}_v$ | $\alpha y + J_4$ | $J_5$ | $\mathbf{J}_H$ |
| 2° | $R$ | $\mathbf{J}_v$ | $\alpha\psi + \beta z + J_4$ | $J_5$ | $\mathbf{J}_H$ |
| 3° | $\psi$ | $\mathbf{J}_v$ | $\alpha \ln R + J_4$ | $R^{-2} J_5$ | $R^{-1}\mathbf{J}_H$ |
| 4° | $q_1$ | $\mathbf{J}_v$ | $\alpha q_2 + J_4$ | $e^{2b_2 q_2} J_5$ | $e^{-b_2 q_2}\mathbf{J}_H$ |
| 5° | $\theta$ | $\mathbf{J}_v$ | $\alpha \ln r + \beta\psi + J_4$ | $r^{-2} J_5$ | $r^{-1}\mathbf{J}_H$ |

Here $\mathbf{V}$ is the particle velocity vector, $\varphi$ is the scalar potential, $\rho$ is the space charge density, $\mathbf{H}$ is the external magnetic field strength vector, $\mathbf{J}_v = \{J_1, J_2, J_3\}$, $J_4$, $J_5$, $\mathbf{J}_H = \{J_6, J_7, J_8\}$ are functions of $\xi$, $\alpha$, $\beta$ are arbitrary constants one of which is always nonzero, x, y, and z are Cartesian coordinates, R, $\psi$, and z are cylindrical coordinates, $q_1$, $q_2$, z are spiral cylindrical coordinates, and r, $\vartheta$, $\psi$, are spherical coordinates. The spiral coordaintes $q_1$, $q_2$ are related to the ordinary polar coordinates R, $\psi$ by

$$q_1 = \frac{b_1}{b_1^2 + b_2^2}\left(\ln R - \frac{b_2}{b_1}\psi\right),$$

$$q_2 = \frac{b_2}{b_1^2 + b_2^2}\left(\ln R + \frac{b_1}{b_2}\psi\right),$$

$$(b_1,\ b_2 = \text{const}).$$

*Below, for brevity, we shall use the term "T-regime emission" to define temperature-limited emission.

As in [3], the flow will be considered regular if the generalized momentum P of the particle is a potential vector. In this case we have the energy integral for the flow as a whole:

$$V^2 + 2\eta\varphi = \text{const}. \tag{1.1}$$

The solutions considered in [1, 2] may correspond to both regular and irregular flows. The solutions given in Table 1 cannot in principle describe regular beams, since the form of the velocity vector and scalar potential does not permit the energy integral to exist in the form (1.1). Moreover, the solutions in Table 1 are invariant. Below, we use the dimensionless variables of [1, 2] unless otherwise stated.

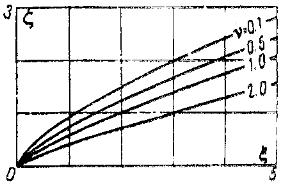


Fig. 1

Table 2 gives external magnetic fields which can be realized without special sustaining devices in the beam ($H_{xi}$ are the physical components of the field strength).

Here $H_{01}$, $H_{02}$, $H_{03}$ are arbitrary constants.

Note that solutions 2° for $\alpha \neq 0$ and 5° for $\beta \neq 0$, which depend upon $\psi$, have meaning only when $0 \leq \psi < 2\pi$. It is clear that the solutions in Table 1 can describe flow in certain diodes with emitter in the form of: 1° a plane x = const; 2° a cylinder R = const, 3° a half-plane $\psi$ = const; 4° a spiral cylinder $q_1$ = const; 5° a cone $\vartheta$ = const, where the emitter potential varies according to a linear or logarithmic law. We can show that there are no solutions of the form 1°–5° for which the normal velocity component and the emitter field simultaneously vanish.

Let us consider in detail flows of types 1° and 2° for $\alpha = 0$. An attempt to satisfy the corresponding total space charge conditions leads to an infinite tangential current density at the emitter. It is shown below that these solutions can describe temperature-limited emission in certain diodes with direct heating.

The system of equations determining the solution 1° for H = 0 has the form

$$uu' = J_4', \qquad \rho u = j, \quad J_4'' = \rho,$$
$$uv' = \alpha \quad (j = \text{const}). \tag{1.2}$$

Here u, v, and w are the velocity components in Cartesian coordinates. The first three equations describe flow in a plane diode, while the last gives the y-component of the velocity. Note that the particles accelerate uniformly in the y-direction. For temperature-limited emission

$$J_4 \sim x,\quad u \sim x^{1/2},\quad \rho \sim x^{-1/2};$$

therefore $v \sim x^{1/2}$, and the current density is everywhere finite. The equipotential surfaces $\varphi$ = const are completely determined by the potential distribution in the corresponding one-dimensional flow

$$\alpha y = -J_4(x) + \text{const.}.$$

Table 2

| | $H_{x^1}$ | $H_{x^2}$ | $H_{x^3}$ |
|---|---|---|---|
| 1° | $H_{01}$ | $H_{02}$ | $H_{03}$ |
| 2° | $H_{01}R^{-1}$ | $H_{02}R^{-1}$ | $H_{03}$ |
| 3° | $H_{01}R^{-1}$ | $H_{02}R^{-1}$ | 0 |
| 4° | $H_{01}R^{-1}$ | $H_{02}R^{-1}$ | 0 |
| 5° | $H_{01}r^{-1}$ | $r^{-1}(H_{01}\operatorname{ctg}\theta + H_{02}\csc\theta)$ | $H_{03}r^{-1}\csc\theta$ |

Thus, in order to find the solution in question it is sufficient to know the quantities describing a plane diode in the T-regime [4]. The solution to this problem can be given in parametric form (Fig. 1):

$$\xi = \frac{1}{2\sqrt{2}}t^3 + \frac{\nu}{2}t^2,\quad \zeta = \frac{1}{2}t^2,\quad \frac{u}{j} = \frac{\sqrt{2J_4}}{j} = \frac{1}{2}t^2 + \frac{\nu\sqrt{2}}{3}t, \tag{1.3}$$

$$\left(\xi = \frac{3x}{j\sqrt{2}},\ \zeta = \frac{y - y_0}{\alpha},\ \nu = \frac{3\varepsilon_0}{j\sqrt{2}}\right).$$

Eliminating t from the expressions for $\xi$ and $\zeta$, we obtain the equation of the trajectory

$$\xi = \zeta^{3/2} + \nu\zeta. \tag{1.4}$$

Curves of $J_4/j^2$ determining the shape of the collector as a function of $\xi$ for different $\nu$ are given in Fig. 2. The particles leave the emitter x = 0 at an angle $\vartheta_0 = \operatorname{arc\,tg}(\varepsilon_0/\alpha)$, where $\varepsilon_0 = J_4'(0)$.

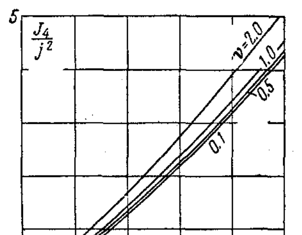


Fig. 2

Let us consider three cases of motion of particles in a uniform magnetic field **H**. We introduce the dimensionless variables $r°$, $V°$, $\varphi°$, $\rho°$ using the formulas

$$r = ar°,\ V = \sqrt{a\eta\varepsilon_0}V°,\ \varphi = -a\varepsilon_0\varphi°,\ \rho = (j/\sqrt{a\eta\varepsilon_0})\rho°.$$

Let the magnetic field initially be directed along the z axis. Then, omitting the dimensionless symbol, we have

$$x^{\cdot\cdot} = J_4' + \omega y^{\cdot},\quad y^{\cdot\cdot} = \beta - \omega x^{\cdot},\quad \rho x^{\cdot} = 1,$$
$$J_4'' = \gamma\rho \tag{1.5}$$
$$\left(\omega = \frac{H}{c}\left(\frac{a\eta}{\varepsilon_0}\right)^{1/2},\ \beta = \frac{\alpha}{a\varepsilon_0},\ \gamma = \frac{4\pi aj}{\varepsilon_0\sqrt{a\eta\varepsilon_0}}\right).$$

Here, the dots and bars are used to denote differentiation with respect to t and x, respectively. We can write the solution of (1.5) in parametric form. Using t as the parameter [5], we easily obtain

$$X = \omega^2 x = 1 - \cos\tau + \mu(\tau - \sin\tau),$$
$$Y = \omega^2(y - y_0) = -\tau + \sin\tau + \mu(1 - \cos\tau) - \lambda\tau^2,$$
$$\Phi = \omega^2 J_4 = 1 - \cos\tau + \mu(\tau - \sin\tau) + \tag{1.6}$$
$$+\lambda\{2(\sin\tau - \tau\cos\tau) + \mu[2(1 - \cos\tau - \tau\sin\tau) + \tau^2]\},$$
$$(\mu = 2\lambda + \beta,\ \lambda = \gamma/2\omega,\ \tau = \omega t).$$

Figure 3 describes the trajectories for different $\mu$, $\lambda$ in X, Y coordinates, while Fig. 4 gives the curves $\Phi = \Phi(X;\mu,\lambda)$. If $\beta = 0$, Eqs. (1.6) define a plane magnetron with temperature-limited emission. The trajectories and curves of $\Phi = \Phi(X;\lambda)$ are shown in Figs. 5 and 6.

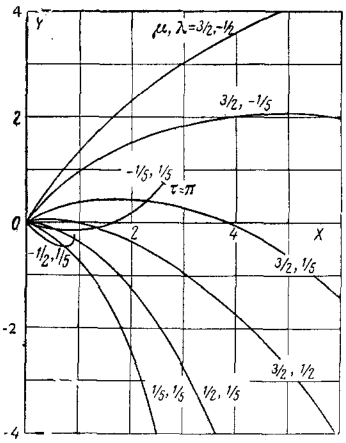


Fig. 3

When the magnetic field is directed along the y axis, we arrive at the equations

$$uu' = J_4' - \omega w,\quad w' = \omega,\quad \rho u = 1,$$
$$J_4'' = \gamma\rho,\quad y^{\cdot\cdot} = \beta. \tag{1.7}$$

The first four equations define a plane magnetron.

The trajectory of a particle moving in this magnetron for T-regime emission will be developed in space as a result of superimposition of the uniform acceleration $y = {}^1/_2\beta t^2$.

If, finally, we equate $H_y$ and $H_z$ to zero, we obtain a superposition of independent motions in the x-direction (in accordance with the same law as for a plane diode)

$$x^{\cdot\cdot} = J_4',\quad \rho x^{\cdot} = 1,\quad J_4'' = \gamma\rho,$$

while in the yz plane (motion in uniform electric and magnetic fields)

$$y^{\cdot\cdot} = \beta + \omega z^{\cdot},\quad z^{\cdot\cdot} = -\omega y^{\cdot}. \tag{1.8}$$

The solution of (1.8) when $v(0) = w(0) = 0$ will be the cycloids

$$Y = \frac{\omega^2}{\beta}(y - y_0) = 1 - \cos\tau, \qquad Z = \frac{\omega^2}{\beta}(z - z_0) = \tau - \sin\tau. \tag{1.9}$$

Development of curves (1.9) in space is realized according to the law $x = x(\tau)$ of a plane diode. Taking the potential $\varphi$ as a constant, we obtain cylindrical surfaces with generators parallel to the z axis, their shape being given by the potential distribution $\varphi = \varphi(x;\ \gamma)$ in a plane diode.

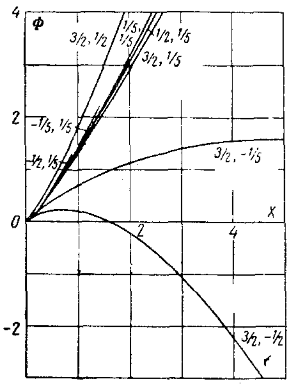


Fig. 4

The system of equations determining the electrostatic solution 2° for $\alpha = 0$ has the form

$$v_R v_R' = J_4', \quad R\rho v_R = j, \qquad R^{-1}(RJ_4')' = \rho, \quad v_R v_z' = \beta. \tag{1.10}$$

The first three equations describe flow in a cylindrical diode, while the last one gives the z-component of the velocity. Consequently, the solution for T-regime emission from an indirectly heated cylinder R = = const is completely determined by the parameters characterizing the corresponding flow in a cylindrical diode [6]. Thus, for example, the collector is obtained by rotating the curve $\varphi(R)$ about the z axis (Fig. 7).* The particles move in planes $\psi$ = const which pass through the emitter axis.

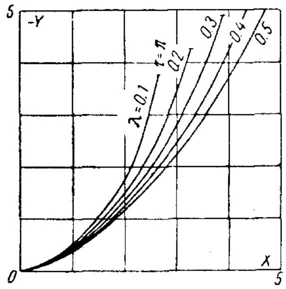


Fig. 5

*In Fig. 7, $\gamma = (\frac{9}{4}\sqrt{2|\eta|}\,R_0|j|)^{-1/3}J_4'(R_0)$, where $R_0$ is the emitter radius.

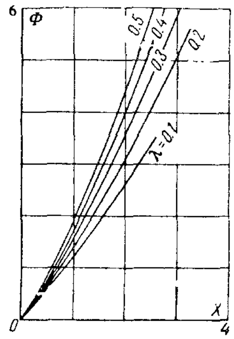


Fig. 6

The system of equations for a beam in a uniform magnetic field directed along the z axis is

$$v_R v_R' - R^{-1}v_\psi^2 = J_4' + Hv_\psi, \quad v_\psi' + R^{-1}v_\psi = -H, \quad R\rho v_R = j, \quad R^{-1}(RJ_4')' = \rho, \quad v_R v_z' = \beta. \tag{1.11}$$

It is clear that the solution of the problem can be given if we know the solution for a cylindrical magnetron in the T-regime.

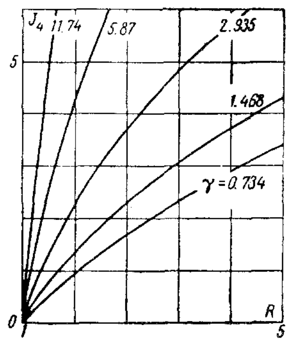


Fig. 7

Consider flow of type 5° assuming $v_\theta = 0$ in the magnetic field given in the last column of Table 2:

$$v_\psi^2 - H_{03}\csc\theta\, v_\psi + \alpha = 0, \quad v_r(v_\psi - H_{01}\,\mathrm{ctg}\,\theta - H_{02}\csc\theta) = \beta\csc\theta, \quad J_4' = -\mathrm{ctg}\,\theta\, v_\psi^2 - H_{01}v_\psi + H_{03}\csc\theta\, v_r, \quad J_5 = \alpha + \csc\theta(\sin\theta J_4')'. \tag{1.12}$$

When the inequality $H_{03}^2 \geqslant 4\alpha$ is satisfied, we have two different solutions where all the determining quantities can be found from (1.12) using algebra and differentiation: only the potential is computed by taking quadratures. When $\beta \neq 0$, this solution is meaningful for $0 \leq \psi < 2\pi$. The particles move in space along spirals, obtained by the intersection of the equipotential surfaces and the cones. Figure 8 gives a schematic representation of the upper half of the equipotential surface $\varphi$ = const. The surfaces $\varphi$ = const intersect the horizontal plane $\theta = \pi/2$ along spirals determined by the equation

$$S(r, {}^1/_2\pi, \psi) = \alpha\ln r + \beta\psi + J_4({}^1/_2\pi) = \text{const}.$$

The trace of the equipotential surface in the half-plane $\psi = \psi_0$ is given by

$$\Phi(r, \theta, \psi_0) = \alpha\ln r + J_4(\theta) + \beta\psi_0 = \text{const}.$$

When $v_r = 0$, $\beta = 0$ the trajectories will be circles located on the equipotential surfaces (surfaces of revolution)

$$\Phi(r, \theta) = \text{const}.$$

Note that the solutions considered define Brillouin flows, since the particles move without intersecting the equipotential surfaces; they belong to the class of generalized Brillouin flows introduced in [7, 8] and characterized by the conservation of generalized momentum along the trajectory.

The solutions given in Table 1 can also determine flows at relativistic speeds. Since there always exists a nonrelativistic region near the emitter, all the above remarks about the conditions at the emitter remain in effect. The solutions 1° and 2° for $\alpha$ = const in this case describe flows from the plane x = const and the cylinder R = const for a linear voltage drop across the emitting surface and for T-regime emission with or without an external magnetic field. However, the construction of each of these solutions is an independent problem and does not reduce to the corresponding one-dimensional problem. Here, of course, it is assumed that definite restrictions are imposed on the relative dimensions of the beam [9, 10], so that we can ignore the magnetic self-field.

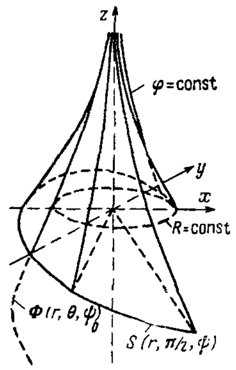


Fig. 8

§2. Consider a flow of type 5° when $\alpha = \beta = 0$, assuming that $v_\theta = 0$ and the velocities are nonrelativistic; the flow takes place in an external magnetic field determined by the last column of Table 2. It is described by the equations

$$\varphi' = H_{03} \csc\theta\, v_r - H_{01} v_\psi - \operatorname{ctg}\theta\, v_\psi^2,$$
$$v_\psi = J_7 = H_{01} \operatorname{ctg}\theta + H_{02} \csc\theta, \qquad (2.1)$$
$$J_5 = \varphi'' + \operatorname{ctg}\theta\, \varphi'.$$

Clearly, system (2.1) is subdefinite, which permits us to take an arbitrary function of $\theta$ as the $v_r$. For $H_{03} = 0$ the potential is independent of $v_r$. Formulas (2.1), in general, specify irregular flow. By requiring the energy integral to exist in the form (1.1) we obtain an equation for $v_r$:

$$v_r v_r' - H_{03} \csc\theta\, v_r + v_\psi v_\psi' + \operatorname{ctg}\theta\, v_\psi^2 + H_{01} v_\psi = 0.$$

Solving this equation, we obtain

$$v_r = H_{03} \ln \operatorname{tg} {}^1/_2\,\theta + v_{r0}. \qquad (2.2)$$

Space spirals on the cones $\theta$ = const will represent the particle trajectories for (2.1) (Fig. 9).

When $v_r = 0$ the trajectories degenerate into circles. It is clear from (2.1) that $v_\psi = 0$ when $\theta = \theta_0$ if $\cos\theta_0 = -H_{02}/H_{01}$, $|H_{02}/H_{01}| \leqslant 1$. By choosing $v_{r0}$ in (2.2) we can achieve the equality $v_r = 0$ at $\theta = \theta_0$. Here, the field will also be zero on the cone $\theta = \theta_0$.

When $v_\psi = 0$, the radial velocity of the regular flow is given by (2.2). If we set $v_{r0} = -H_{03} \ln \operatorname{tg} {}^1/_2\,\theta_0$, total space charge conditions* will be satisfied on the cone $\theta = \theta_0$.

Let us now examine Brillouin flow in the r-direction in a conical diode at relativistic velocities, where strict allowance is made for the magnetic self-field. The corresponding system (S/H) is subdefinite:

$$\varphi' = v_r J_8, \quad \csc\theta\,(\sin\theta\,\varphi')' = J_5,$$
$$\csc\theta\,(\sin\theta\, J_8)' = v_r J_5, \qquad (2.3)$$
$$v_\theta = v_\psi = J_6 = J_7 = 0.$$

Therefore we can take any function satisfying the inequality $|v_r| < 1$ as the radial velocity (the velocities are referred to the speed of light). Then

$$\varphi' = \frac{a}{\sin\theta} \frac{v_r}{\sqrt{1 - v_r^2}}, \qquad J_5 = \frac{a}{\sin\theta}\left(\frac{v_r}{\sqrt{1 - v_r^2}}\right)',$$
$$J_8 = \frac{a}{\sin\theta} \frac{1}{\sqrt{1 - v_r^2}}. \qquad (2.4)$$

For arbitrary $v_r$ formulas (2.4) describe vortex flow. By requiring that there exist the relativistic equivalent of (1.1), we arrive at the following expressions for the regular flow:

$$v_r = \pm \frac{1 - b^2 (\operatorname{tg} {}^1/_2\,\theta)^{2a}}{1 + b^2 (\operatorname{tg} {}^1/_2\,\theta)^{2a}}, \quad \varphi = \frac{1 + b^2 (\operatorname{tg} {}^1/_2\,\theta)^{2a}}{2b (\operatorname{tg} {}^1/_2\,\theta)^{a}},$$
$$J_8 = \mp \frac{a}{\sin\theta}\,\varphi. \qquad (2.5)$$

Here, $a$ and b are arbitrary constants.

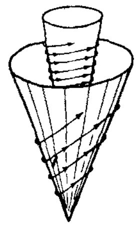

Fig. 9

*Article [11] dealt with the problem involving the compensation of the space charge forces of a conical beam by a magnetic field when $H_{01} = -H_{02}$, $H_{03} = 0$. The assumption that the radial current density was independent of $\theta$ led to an approximate solution which is valid for small angles of convergence. Formulas (2.1) and (2.2) determine the exact solution which exists for any $\theta$ in the more general case: no restrictions on $H_1$, $H_2$, and $H_3$ and $v_r \neq$ const.

When $a = b = 1$, formulas (2.5) have the very simple form

$$v_r = \pm\cos\theta, \quad v_\theta = v_\psi = 0,$$
$$\varphi = \csc\theta, \quad \rho = r^{-2}\csc^3\theta, \tag{2.6}$$
$$H_r = H_\theta = 0, \quad H_\psi = \mp r^{-1}\csc^2\theta.$$

For each pair of values of $a$ and b there exists a cone $\theta = \theta_0$ with potential $\varphi = 1$, on which total space charge conditions hold,

$$\operatorname{tg} {}^1/_2\,\theta_0 = \left(\frac{1}{b}\right)^{1/a}.$$

For (2.6) we have $\theta_0 = \pi/2$. We note in conclusion that flow in the $\psi$-direction can only be regular.

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